

COMPLETE TRANSLATING SOLITONS TO THE MEAN CURVATURE FLOW IN \mathbb{R}^3 WITH NONNEGATIVE MEAN CURVATURE

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ABSTRACT. We prove that any complete immersed two-sided mean convex translating soliton $\Sigma \subset \mathbb{R}^3$ for the mean curvature flow is convex. As a corollary it follows that an entire mean convex graphical translating soliton in \mathbb{R}^3 is the axisymmetric “bowl soliton”. We also show that if the mean curvature of Σ tends to zero at infinity, then Σ can be represented as an entire graph and so is the “bowl soliton”. Finally we classify all locally strictly convex graphical translating solitons defined over strip regions.

1. INTRODUCTION

A complete immersed hypersurface $f : \Sigma^n \rightarrow \mathbb{R}^{n+1}$ with trivial normal bundle (two-sided for short) is called a translating soliton for the mean curvature flow, with respect to a unit direction e_{n+1} , if its mean curvature is given by $H = \langle N, e_{n+1} \rangle$ where N is a global unit normal field for Σ . Then $F(x, t) := f(x) + te_{n+1}$ satisfies

$$\Delta^\Sigma F = HN = \langle N, e_{n+1} \rangle N = (e_{n+1})^\perp = F_t^\perp,$$

thus justifying the terminology. Translating solitons form a special class of eternal solutions for the mean curvature flow that besides having their own intrinsic interest, are models of slow singularity formation. Therefore there has been a great deal of effort in trying to classify them in the case $H > 0$. In this paper, *we shall always assume our translating solitons are mean convex which by abuse of language we take to mean $H > 0$* . The abundance of glueing constructions for translating solitons with high genus and H changing sign (see [19],[20],[21], [9], [7], [23]) suggests a general classification is unlikely.

For $n = 1$ the unique solution is the grim reaper curve $\Gamma : x_2 = \log \sec x_1, |x_1| < \frac{\pi}{2}$, while for $n \geq 2$ we have the one parameter family of convex grim cylinders (see Lemma 5.1 for the $n = 2$ case)

$$(1.1) \quad x_{n+1} = \lambda^2 \log \sec \frac{x_1}{\lambda} + \sum_{k=2}^n \alpha_k x_k, \quad \sum_{k=2}^n \alpha_k^2 = \lambda^2 - 1, \quad |x_1| < \frac{\pi}{2} \lambda, \quad \lambda \geq 1,$$

Research of the first author is partially supported in part by the NSF.

which can be obtained from the standard grim cylinders $\Gamma \times \mathbb{R}^{n-1}$ by a rotation and scaling. The family of grim graphical solitons in \mathbb{R}^3 :

$$(1.2) \quad u^\lambda(x_1, x_2) = \lambda^2 \log \sec \frac{x_1}{\lambda} \pm Lx_2, \quad L = \sqrt{\lambda^2 - 1}, \quad \lambda \geq 1$$

defined over the strip $\mathcal{S}^\lambda := \{(x_1, x_2) : |x_1| < R := \lambda \frac{\pi}{2}\}$ will play a central role in our classification of mean convex graphical translating solitons in \mathbb{R}^3 .

Similarly if $x_{m+1} = v(x_1, \dots, x_m)$ is a graphical translating soliton in \mathbb{R}^{m+1} , then for $x = (x_1, \dots, x_n) = (x', x_{m+1}, \dots, x_n)$,

$$(1.3) \quad v^\lambda(x) := \lambda^2 v\left(\frac{x'}{\lambda}\right) + \sum_{k=m+1}^n \alpha_k x_k, \quad \sum_{k=m+1}^n \alpha_k^2 = \lambda^2 - 1, \quad \lambda \geq 1,$$

is a graphical translating soliton in $\mathbb{R}^{n+1} = \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}$. There is as well a unique (up to horizontal translation) axisymmetric solution called the “bowl soliton” [1], [4] which has the asymptotic expansion as an entire graph $u(x) = \frac{1}{2(n-1)}|x|^2 - \log|x| + O(1)$.

White [30] tentatively conjectured that all convex translating solutions have the form

$$\{(x, y, z) \in \mathbb{R}^j \times \mathbb{R}^{n-j} \times \mathbb{R} : z = f(|x|)\}$$

for $j \geq 2$; for $j = 1$, f defines the grim reaper curves so is defined on an interval. White remarks that even if this conjecture is false, *it may be true for blow up limits of mean convex mean curvature flows*. In [28] Wang proved that in dimension $n = 2$, any entire convex graphical translating soliton must be rotationally symmetric, and hence the bowl soliton. He also showed there exist entire locally strictly convex graphical translating solitons for dimensions $n > 2$ that are not rotationally symmetric (thus disproving one conjecture of White [30]) as well as complete locally strictly convex graphical translating solitons defined over strip regions in \mathbb{R}^n . Wang also conjectured that for $n = 2$, any entire graphical translating soliton must be locally strictly convex and a similar statement in dimension $n > 2$ under the additional assumption that H tends to zero at infinity.

More recently, Haslhofer [12] proved the uniqueness of the bowl soliton in arbitrary dimension under the assumption that the translating soliton Σ is α -noncollapsed and uniformly 2-convex. The α -noncollapsed condition means that for each $P \in \Sigma$, there are closed balls B^\pm disjoint from $\Sigma - P$ of radius at least $\frac{\alpha}{H(P)}$ with $B^+ \cap B^- = \{P\}$. It figures prominently in the regularity theory for mean convex mean curvature flow [29], [30], [15], [25]. The 2-convex condition (automatic if $n = 2$) means that if $\kappa_n \leq \kappa_{n-1} \leq \dots \kappa_1$ are the ordered principal curvatures of Σ , then $\kappa_n + \kappa_{n-1} \geq \beta H$ for some uniform $\beta > 0$. The α -noncollapsed condition is a deep and powerful property of weak solutions of the mean

convex mean curvature flow [30],[13] which implies that any complete (α -noncollapsed) mean convex translating soliton Σ is convex with uniformly bounded second fundamental form.

The main result of this paper is a proof of a more general form of the $n = 2$ conjecture of Wang.

Theorem 1.1. *Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton for the mean curvature flow with nonnegative mean curvature. Then Σ is convex.*

By Sacksteder's theorem [23], the condition $H > 0$ and Corollary 2.1 of [28], we may conclude

Corollary 1.2. *Σ is the boundary of a convex region in \mathbb{R}^3 whose projection on the plane spanned by e_1, e_2 is (after rotation of coordinates) either a strip region $\{(x_1, x_2) : |x_1| < R\}$ or \mathbb{R}^2 . Moreover Σ is the graph of a function $u(x_1, x_2)$ which satisfies the equation*

$$(1.4) \quad \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \frac{1}{\sqrt{1 + |\nabla u|^2}},$$

or in nondivergence form,

$$(1.5) \quad (1 + u_{x_2}^2)u_{x_1x_1} - 2u_{x_1}u_{x_2}u_{x_1x_2} + (1 + u_{x_1}^2)u_{x_2x_2} = 1 + u_{x_1}^2 + u_{x_2}^2.$$

Combining Theorem 1.1 with Theorem 1.1 in [28] we have

Corollary 1.3. *Any entire solution in \mathbb{R}^2 to the equation*

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \frac{1}{\sqrt{1 + |\nabla u|^2}}$$

must be rotationally symmetric in an appropriate coordinate system and hence is the bowl soliton.

A necessary and sufficient condition for translating solitons to be graphical over \mathbb{R}^2 and thus the bowl soliton is given in the next theorem.

Theorem 1.4. *Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton for the mean curvature flow with nonnegative mean curvature and suppose that $H(P) \rightarrow 0$ as $P \in \Sigma$ tends to infinity. Then Σ is after translation the axisymmetric bowl soliton.*

The existence of the grim family u^λ (see 1.2), the convexity Theorem 1.1 and a global curvature bound (see Theorem 2.8 in section 2) are the key tools that allows us to classify locally strictly convex graphical translating solitons defined over strips.

Theorem 1.5. *Let $\Sigma = \text{graph}(u)$ be a complete locally strictly convex translating soliton defined over a strip region $\mathcal{S}^\lambda := \{(x_1, x_2) : |x_1| < R := \lambda \frac{\pi}{2}\}$. Then (after possibly relabeling the e_2 direction)*

i. *For $\lambda \leq 1$ there is no locally strictly convex solution in \mathcal{S}^λ .*

ii. $\lim_{x_2 \rightarrow +\infty} u_{x_2}(x_1, x_2) = L := \sqrt{\lambda^2 - 1}$, $\lambda > 1$.

iii. $\lim_{x_2 \rightarrow -\infty} u_{x_2}(x_1, x_2) = -L$.

iv.

$$(1.6) \quad \begin{aligned} \lim_{A \rightarrow \pm\infty} (u(x_1, x_2 + A) - u(0, A)) &= u^\lambda(x_1, x_2) \\ \lim_{A \rightarrow \pm\infty} u_{x_1}(x_1, A) &= \lambda \tan \frac{x_1}{\lambda}. \end{aligned}$$

The above limits are uniform in $|x_1| \leq R - \varepsilon$ for any $\varepsilon > 0$.

v. *For $P = (x_1, x_2, u(x_1, x_2)) \in \Sigma$, $(x_1, x_2) \in \mathcal{S}^\lambda$,*

$$(1.7) \quad H(P) \leq R - |x_1|, \quad H(P) \geq \theta(\varepsilon) \text{ if } |x_1| \leq R - \varepsilon.$$

vi. $u(x_1, x_2) = u(-x_1, x_2)$ and $u_{x_1}(x_1, x_2) > 0$ for $x_1 > 0$.

vii. *For any $\lambda > 1$ there exists a complete locally strictly convex translating soliton $\Sigma = \text{graph}(u)$ defined over \mathcal{S}^λ . Moreover, $u(x_1, x_2) = u(-x_1, x_2)$, $u(x_1, x_2) = u(x_1, -x_2)$.*

Remark 1.6. At the moment it is not clear if the solution obtained in Theorem 1.5 part vii. is unique.

The organization of the paper is as follows. In section 2 we show that any immersed two-sided mean convex translating soliton is stable. This allows us to use the method of Choi-Schoen [3] as modified by Colding-Minicozzi [5], [6] to prove a global curvature bound (Theorem 2.8). This is needed for the proof of Theorem 1.1 in section 3, which is based on a delicate maximum principle argument. In section 4 we give the proof of Theorem 1.4. This result also allows us to start the proof of Theorem 1.5 which is long and detailed. The proof of parts i.-vi. is contained in section 5 and the proof of existence part vii. by a direct construction is contained in section 6.

2. STABILITY AND CURVATURE ESTIMATES.

We will need the the following well-known identities that hold on any translating soliton in \mathbb{R}^{n+1} (see for example [18]).

Lemma 2.1. *Let Σ be a two-sided immersed translating soliton in \mathbb{R}^{n+1} with second fundamental form A . Let $A = (h_{ij})$ be the second fundamental form of Σ , $u = x_{n+1}|_{\Sigma}$ and $\Delta^f := \Delta^{\Sigma} + \langle \nabla, e_{n+1} \rangle$ be the drift Laplacian on Σ . Then*

- i. $|\nabla u|^2 = 1 - H^2$, $\Delta^{\Sigma} u = H^2$
- ii. $\Delta^f A + |A|^2 A = 0$,
- iii. $\Delta^f H + H|A|^2 = 0$,
- iv. $\Delta^f(|A|)^2 - 2|\nabla A|^2 + 2|A|^4 = 0$.

It is well known that a translating soliton Σ with respect to the direction e_{n+1} in \mathbb{R}^{n+1} is a critical point of the weighted area functional

$$\tilde{\mathcal{A}}(\Sigma) = \int_{\Sigma} e^{x_{n+1}} dv$$

and is in fact minimal with respect to the weighted Euclidean metric $e^{x_{n+1}}\delta$ on \mathbb{R}^{n+1} . The second variation of $\tilde{\mathcal{A}}$ with respect to a compactly supported normal variation ηN is easily computed to be

$$\tilde{\mathcal{A}}''(0) = \int_{\Sigma} (|\nabla \eta|^2 - |A|^2 \eta^2) e^{x_{n+1}} dv = \int_{\Sigma} -\eta L \eta e^{x_{n+1}} dv ,$$

where $L\eta = e^{-x_{n+1}} \operatorname{div}^{\Sigma}(e^{x_{n+1}} \nabla \eta) + |A|^2 \eta$ is the associated stability operator.

Proposition 2.2. *i. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a complete immersed two-sided translating soliton with respect to e_{n+1} with $H \geq 0$. Then Σ is strictly stable, that is, $\lambda_1(-L)(D) > 0$ on any compact subdomain $D \subset \Sigma$.*

ii. The following are equivalent:

- a. *There exists a positive solution v of $Lv = 0$ for every bounded D .*
- b. $\lambda_1(-L)(D) \geq 0$ for every bounded D .
- c. $\lambda_1(-L)(D) > 0$ for every bounded D .

Proof. i. We may assume that $H > 0$. Then $w = \log H$ satisfies $\Delta w + \langle \nabla w, e_{n+1} \rangle + |A|^2 = -|\nabla w|^2$. Hence

$$\begin{aligned} \int_{\Sigma} (\eta^2 |A|^2 - |\nabla \eta|^2) e^{x_{n+1}} dv &= - \int_{\Sigma} (\eta^2 |\nabla w|^2 - 2\eta \langle \nabla \eta, \nabla w \rangle + |\nabla \eta|^2) e^{x_{n+1}} dv \\ &= - \int_{\Sigma} |\eta \nabla w - \nabla \eta|^2 e^{x_{n+1}} dv < 0. \end{aligned}$$

ii. The proof is a straightforward modification of that of Fischer-Colbrie and Schoen [8] and will not be given. \square

We will need the following corollary for the case at hand $n = 2$.

Corollary 2.3. *Let $\mathcal{B}_\rho(P)$ be an intrinsic ball in Σ with $\rho < 2\pi$. Then $\mathcal{B}_\rho(P)$ is disjoint from the conjugate locus of P and*

$$(2.1) \quad \int_{\Sigma} f^2 |A|^2 dv \leq e^{2\rho} \int_{\Sigma} |\nabla f|^2 dv .$$

for $f \in H_0^1(\mathcal{B}_\rho(P))$.

Proof. Since $K \leq \frac{H^2}{4} \leq \frac{1}{4}$, the first statement follows from standard comparison geometry. Also $|\nabla x_3|^2 = 1 - H^2 \leq 1$, so $|x_3(Q) - x_3(P)| \leq \rho$ for any $Q \in \mathcal{B}_\rho(P)$. Hence $e^{-\rho} e^{x_3}(P) \leq e^{x_3}(Q) \leq e^{\rho} e^{x_3}(P)$ and the result follows from Proposition 2.2. \square

We now follow the Colding-Minicozzi method [5, 6] with appropriate modification, to prove intrinsic area bounds and then curvature bounds. For two dimensional graphs, such curvature estimates follow immediately from the work of Leon Simon [27], see also [24].

Proposition 2.4. *Let $\Sigma \subset \mathbb{R}^3$ be a two-sided immersed translating soliton with $H \geq 0$ and let $\mathcal{B}_\rho(P)$ be a topological disk in Σ . Then $\mathcal{B}_\rho(P)$ is disjoint from the cut locus of P for $e^{2\rho} < 2$ and*

$$(2.2) \quad \begin{aligned} i. & \quad \frac{A(\mathcal{B}_\rho(P))}{\rho^2} \leq 2\pi , \\ ii. & \quad \int_{\mathcal{B}_{\mu^2\rho}(P)} |A|^2 dv \leq 2\pi \{ (\log \frac{1}{\mu})^{-2} + 2(\log \frac{1}{\mu})^{-1} \} \text{ for } \mu \in (0, 1). \end{aligned}$$

Proof. We first prove $\mathcal{B}_\rho(P)$ is disjoint from the cut locus $C(P)$ of P , that is the injectivity radius of P satisfies $r_0 := \text{inj}_P(\Sigma) > \rho$. Suppose for contradiction that $Q \in \partial\mathcal{B}_{r_0}(P)$ is a cut locus of P and $r_0 \leq \rho$. We know by Corollary 1.2 that Q is not in the conjugate locus of P so by Klingenberg's lemma (see for example Lemma 5.7.12 of [22]) there are two minimizing geodesics from P to Q which fit together smoothly at Q with a possible corner at P and bounding a domain $D \subset \mathcal{B}_{r_0}(P)$. By Gauss-Bonnet,

$$\frac{1}{4}A(D) \geq \int_D K dv = 2\pi - \int_{\partial D} \kappa_g d\sigma \geq \pi ,$$

hence $A(\mathcal{B}_{r_0}(P)) > A(D) \geq 4\pi$. On the other hand by (2.2) with $\rho = r_0$ (proved below), $A(\mathcal{B}_{r_0}(P)) < 2\pi r_0^2$. Therefore $r_0 > \sqrt{2}$, contradicting $r_0 \leq \rho < \frac{1}{2} \log 2$.

We now prove the stated inequalities. Let $l(s)$ be the length of $\partial\mathcal{B}_s(P)$ and $K(s) = \int_{\mathcal{B}_s(P)} K \, dv$. By Gauss-Bonnet,

$$(2.3) \quad l'(s) = \int_{\partial\mathcal{B}_s(P)} \kappa_g \, d\sigma = 2\pi - K(s) .$$

For $r = d(P, x)$, $f(x) = \eta(r)$, $\eta, -\eta' \geq 0$, $\eta(\rho) = 0$, we use the stability inequality (2.1) and write

$$|A|^2 = H^2 - 2K \geq -2K,$$

to obtain (for $\rho \leq r_0$)

$$(2.4) \quad -2 \int_{\mathcal{B}_\rho(P)} K f^2 \leq 2 \int_{\mathcal{B}_\rho(P)} |\nabla f|^2$$

By the coarea formula,

$$(2.5) \quad \begin{aligned} \int_{\mathcal{B}_\rho(P)} K f^2 &= \int_0^\rho \eta^2(s) \int_{\partial\mathcal{B}_s(P)} K \, d\sigma \, ds = \int_0^\rho \eta^2(s) K'(s) \, ds \\ \int_{\mathcal{B}_\rho(P)} |\nabla f|^2 &= \int_0^\rho \int_{\partial\mathcal{B}_s(P)} |\nabla f|^2 \, d\sigma \, ds = \int_0^\rho (\eta')^2(s) l(s) \, ds \end{aligned}$$

For $\eta(r) = 1 - \frac{r}{\rho}$, using (2.3)-(2.5) this gives

$$(2.6) \quad -\frac{4}{\rho} \int_0^\rho (2\pi - l'(s)) \left(1 - \frac{s}{\rho}\right) ds \leq 2 \frac{A(\mathcal{B}_\rho(P))}{\rho^2}$$

By integration by parts,

$$(2.7) \quad \int_0^\rho (2\pi - l'(s)) \left(1 - \frac{s}{\rho}\right) ds = \pi\rho - \frac{A(\mathcal{B}_\rho(P))}{\rho}$$

Finally combining (2.6), (2.7) and simplifying gives inequality i. in (2.2) For part ii. we use the logarithmic cutoff

$$(2.8) \quad \eta(s) = \begin{cases} 1 & \text{if } s \leq \mu^2 \rho \\ \frac{\log \frac{s}{\rho}}{\log \mu} - 1 & \text{if } \mu^2 \rho < s < \mu \rho \\ 0 & \text{if } s > \mu \rho \end{cases}$$

in (2.1). Then

$$\begin{aligned}
 (2.9) \quad \int_{\mathcal{B}_{\mu^2\rho}(P)} |A|^2 &\leq \frac{2}{(\log \mu)^2} \int_{\mu^2\rho}^{\mu\rho} \frac{l(s)}{s^2} ds \\
 &\leq \frac{2}{(\log \mu)^2} \left\{ \frac{A(\mathcal{B}_s(P))}{s^2} \Big|_{\mu^2\rho}^{\mu\rho} + 2 \int_{\mu^2\rho}^{\mu\rho} \frac{A(\mathcal{B}_s(P))}{s^3} ds \right\} \\
 &\leq 4\pi \left\{ \frac{1}{(\log \mu)^2} + \frac{2}{\log \frac{1}{\mu}} \right\}
 \end{aligned}$$

by (2.2) part i. □

For later use we will need two well known lemmas; the first is an extrinsic mean value inequality (for a proof see [6] p. 26-27) and the second one says that curvature bounds implies graphical with intrinsic and extrinsic balls related (see [6] Lemma 2.4).

Lemma 2.5. *Let $\Sigma \subset \mathbb{R}^3$ be an embedded surface with $x_0 \in \Sigma$, $B_s(x_0) \cap \partial\Sigma = \emptyset$. Suppose the mean curvature of Σ satisfies $|H| \leq C$ and $f \geq 0$ is a smooth function on Σ satisfying $\Delta^\Sigma f \geq -\lambda s^{-2}f$. Then*

$$f(x_0) \leq \frac{e^{(\frac{\lambda}{4} + Cs)}}{\pi s^2} \int_{B_s(x_0) \cap \Sigma} f dv.$$

Lemma 2.6. *Let $\Sigma \subset \mathbb{R}^3$ be an immersed surface with $16s^2 \sup_\Sigma |A|^2 \leq 1$. If $P \in \Sigma$ and $\text{dist}^\Sigma(P, \partial\Sigma) \geq 2s$, then*

- i. $\mathcal{B}_{2s}(P)$ can be written as a graph of a function u over $T_P\Sigma$ with $|\nabla u| \leq 1$ and $\sqrt{2}s|\text{Hess}_u| \leq 1$;
- ii. The connected component of $B_s(P) \cap \Sigma$ containing P is contained in $\mathcal{B}_{2s}(P)$.

Proposition 2.7. (Choi-Schoen type curvature bound) *Let $\Sigma \subset \mathbb{R}^3$ be a two-sided immersed translating soliton and let $\mathcal{B}_\rho(P)$ be disjoint from the cut locus of P . Then there exists $\varepsilon, \tau < \frac{\sqrt{\varepsilon}}{2\pi} < \rho$ such that if for all $r_0 \leq \tau$, there holds $\int_{\mathcal{B}_{r_0}(P)} |A|^2 \leq \varepsilon$, then for all $0 < \sigma \leq r_0$, $y \in \mathcal{B}_{r_0-\sigma}(P)$ we have $|A|^2(y) \leq \sigma^{-2}$.*

Proof. Define $F(x) = (r_0 - r(x))^2 |A|^2(x)$ on $\mathcal{B}_{r_0}(P)$ and suppose F assumes its maximum at x_0 . Note that if $F(x_0) \leq 1$, then $\sigma^2 |A|^2(y) \leq (r_0 - r(y))^2 |A|^2(y) \leq F(x_0) \leq 1$ and we are done. If not, define σ by $4\sigma^2 |A|^2(x_0) = 1$. Then by the triangle inequality on $\mathcal{B}_\sigma(x_0)$,

$$\frac{1}{2} \leq \frac{r_0 - r(x)}{r_0 - r(x_0)} \leq 2,$$

which implies $\sup_{\mathcal{B}_\sigma(x_0)} |A|^2 \leq 4|A|^2(x_0) = \sigma^{-2}$. On Σ we have the Simons' type equation $L(|A|^2) - 2|\nabla A|^2 + 2|A|^4 = 0$ which implies $\Delta(|A|^2 + \frac{1}{2}) \geq -2|A|^2(|A|^2 + \frac{1}{2})$. Hence for $f(x) := |A|^2 + \frac{1}{2}$, $\Delta f \geq -2\sigma^{-2}f$ on $\mathcal{B}_\sigma(x_0)$. Using Lemmas 2.6 2.5 with $s = \frac{\sigma}{4}$, $\lambda = \frac{1}{6}$, $C = 1$ and Proposition 2.4 part i. ($a = 0$), we find

$$(2.10) \quad \left(\frac{1}{4\sigma^2} + \frac{1}{2}\right) = |A|^2(x_0) + \frac{1}{2} \leq \frac{16}{\pi\sigma^2} e^{(\frac{1}{24} + \frac{\sigma}{4})} \{\varepsilon + 2\pi r_0^2\}.$$

Multiplying (2.10) by σ^2 we find

$$(2.11) \quad \frac{1}{4} < \frac{1}{4} + \frac{\sigma^2}{2} \leq \frac{16}{\pi} e^{(\frac{1}{24} + \frac{\sigma}{4})} \{\varepsilon + 2\pi r_0^2\} < \frac{32e}{\pi} \varepsilon$$

which is impossible for $\varepsilon \leq \frac{\pi}{128e}$. \square

We are now in a position to prove the curvature estimate we will need in the next section.

Theorem 2.8. *Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton with respect to e_3 with $H \geq 0$. Then there is a universal constant C such that $|A|^2(P) \leq C$ for $P \in \Sigma$.*

Proof. For any $P \in \Sigma$, we fix $\rho > 0$ such that $e^\rho < 2$ as in Proposition 2.4 so that $\mathcal{B}_\rho(P)$ is disjoint from the cut locus of P . We may assume $\mathcal{B}_\rho(P)$ is a topological disk since by Proposition 2.2, the universal cover of $\mathcal{B}_\rho(P)$ endowed with pull-back metric is also a stable translating soliton with nonnegative mean curvature. Thus using Proposition 2.4 part ii. with $\mu = e^{-\frac{6\pi}{\varepsilon}}$, the conditions of Proposition 2.7 are satisfied for $\tau = \mu^2 \rho$. We can choose $\sigma = \tau$ and obtain $|A|^2(P) \leq \tau^{-2}$. \square

3. PROOF OF THEOREM 1.1.

We restate for the readers convenience our main result.

Theorem 3.1. *Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton for the mean curvature flow with nonnegative mean curvature. Then Σ is convex.*

Proof. Without loss of generality, we assume Σ satisfies

$$H = \langle \nu, e_3 \rangle > 0.$$

Let

$$(3.1) \quad f(x_1, x_2) = \frac{x_1 + x_2}{2} + \left[\left(\frac{x_1 - x_2}{2} \right)^2 \right]^{1/2},$$

$$(3.2) \quad \phi(r) = \begin{cases} r^4 e^{-1/r^2} & \text{if } r < 0 \\ 0 & \text{if } r \geq 0, \end{cases}$$

and

$$(3.3) \quad g(z) = f(z) \sum_{i=1}^2 \phi\left(\frac{z_i}{f(z)}\right).$$

It's easy to see that g is smooth when $z_1 \neq z_2$. Now denote

$$G(A) = g(\kappa(A)) \text{ and } F(A) = f(\kappa(A)),$$

where A is a 2×2 symmetric matrix and $\kappa(A)$ are the eigenvalues of A . Now let A be the second fundamental form of Σ . Then $G/F \leq 1$ and therefore G/F achieves its maximum either at an interior point or “at infinity”. By a straight forward calculation, we have

$$(3.4) \quad \begin{aligned} & \Delta^\Sigma \left(\frac{G}{F} \right) + 2 \left\langle \frac{\nabla F}{F}, \nabla \left(\frac{G}{F} \right) \right\rangle + \left\langle \nabla \left(\frac{G}{F} \right), e_3 \right\rangle \\ &= -\frac{G^{ij} h_{ij} |A|^2}{F} + \frac{GF^{ij} h_{ij} |A|^2}{F^2} + \left(\frac{G^{pq,rs}}{F} - \frac{GF^{pq,rs}}{F^2} \right) h_{pqk} h_{rsk}. \end{aligned}$$

We next compute the last term in (3.4).

$$(3.5) \quad G^{pq,rs} h_{pqk} h_{rsk} = g^{pq} h_{ppk} h_{qqk} + 2 \frac{g^2 - g^1}{\kappa_2 - \kappa_1} h_{12k}^2,$$

where

$$(3.6) \quad \begin{aligned} g^{pq} &= f^{pq} \sum_{i=1}^2 \left[\phi\left(\frac{z_i}{f}\right) - \frac{z_i}{f} \dot{\phi}\left(\frac{z_i}{f}\right) \right] \\ &+ \frac{1}{f} \sum_{i=1}^2 \ddot{\phi}\left(\frac{z_i}{f}\right) \left(\delta_i^p - \frac{z_i}{f} f^p \right) \left(\delta_i^q - \frac{z_i}{f} f^q \right). \end{aligned}$$

It follows that

$$(3.7) \quad \begin{aligned} & FG^{pq,rs} - GF^{pq,rs} \\ &= (fg^{pq} - gf^{pq}) + 2 \left\{ f \frac{g^2 - g^1}{\kappa_2 - \kappa_1} - g \frac{f^2 - f^1}{\kappa_2 - \kappa_1} \right\} \\ &= I + II. \end{aligned}$$

We proceed to calculate the terms I and II.

$$\begin{aligned}
 (3.8) \quad I &= f^{pq} \left[g - \sum_{i=1}^2 z_i \dot{\phi} \left(\frac{z_i}{f} \right) \right] + \sum_{i=1}^2 \ddot{\phi} \left(\frac{z_i}{f} \right) \left(\delta_i^p - \frac{z_i}{f} f^p \right) \left(\delta_i^q - \frac{z_i}{f} f^q \right) - g f^{pq} \\
 &= -f^{pq} \sum_{i=1}^2 z_i \dot{\phi} \left(\frac{z_i}{f} \right) + \sum_{i=1}^2 \ddot{\phi} \left(\frac{z_i}{f} \right) \left(\delta_i^p - \frac{z_i}{f} f^p \right) \left(\delta_i^q - \frac{z_i}{f} f^q \right),
 \end{aligned}$$

$$\begin{aligned}
 (3.9) \quad g^2 - g^1 &= \dot{\phi} \left(\frac{z_2}{f} \right) + f^2 \sum_{i=1}^2 \left[\phi \left(\frac{z_i}{f} \right) - \frac{z_i}{f} \dot{\phi} \left(\frac{z_i}{f} \right) \right] \\
 &\quad - \dot{\phi} \left(\frac{z_1}{f} \right) - f^1 \sum_{i=1}^2 \left[\phi \left(\frac{z_i}{f} \right) - \frac{z_i}{f} \dot{\phi} \left(\frac{z_i}{f} \right) \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (3.10) \quad \frac{\kappa_2 - \kappa_1}{2} II &= f(g^2 - g^1) - g(f^2 - f^1) \\
 &= f \left[\dot{\phi} \left(\frac{z_2}{f} \right) - \dot{\phi} \left(\frac{z_1}{f} \right) \right] + f^2 \left[g - \sum_{i=1}^2 z_i \dot{\phi} \left(\frac{z_i}{f} \right) \right] \\
 &\quad - f^1 \left[g - \sum_{i=1}^2 z_i \dot{\phi} \left(\frac{z_i}{f} \right) \right] - g(f^2 - f^1) \\
 &= f \left[\dot{\phi} \left(\frac{z_2}{f} \right) - \dot{\phi} \left(\frac{z_1}{f} \right) \right] + \sum_{i=1}^2 z_i \dot{\phi} \left(\frac{z_i}{f} \right) (f^1 - f^2).
 \end{aligned}$$

Assume $\kappa_1 > 0 > \kappa_2$; then

$$\begin{aligned}
 (3.11) \quad \Delta^\Sigma \left(\frac{G}{F} \right) &+ 2 \left\langle \frac{\nabla F}{F}, \nabla \left(\frac{G}{F} \right) \right\rangle + \left\langle \nabla \left(\frac{G}{F} \right), e_3 \right\rangle \\
 &= \left(\frac{G^{pq,rs}}{F} - \frac{GF^{pq,rs}}{F^2} \right) h_{pqk} h_{rsk} = \frac{1}{F^2} (FG^{pq,rs} - GF^{pq,rs}) h_{pqk} h_{rsk} \\
 &= \frac{1}{\kappa_1^2} \ddot{\phi} \left(\frac{\kappa_2}{\kappa_1} \right) \left(\delta_2^p - \frac{\kappa_2}{\kappa_1} f^p \right) \left(\delta_2^q - \frac{\kappa_2}{\kappa_1} f^q \right) h_{ppk} h_{qqk} + \frac{2}{\kappa_1^2} \dot{\phi} \left(\frac{\kappa_2}{\kappa_1} \right) \left(\frac{\kappa_1 + \kappa_2}{\kappa_2 - \kappa_1} \right) h_{12k}^2 \geq 0.
 \end{aligned}$$

Here we used $f^{pq} = 0$, $\dot{\phi} \left(\frac{\kappa_1}{f} \right) = 0$, and $f^2 = 0$ when $\kappa_1 > 0 > \kappa_2$. It follows that if G/F achieves its maximum at an interior point, then by the strong maximum principle we have

$\frac{G}{F} \equiv \text{constant}$. If $\kappa_2/\kappa_1 \neq 0$, then $\varphi := \log \frac{|A|^2}{H^2}$ is constant and so

$$\begin{aligned}
 0 &= \nabla \varphi = \frac{\nabla |A|^2}{|A|^2} - 2 \frac{\nabla H}{H} \\
 0 &= \Delta^\Sigma \varphi + \langle \nabla \varphi, e_3 \rangle \\
 (3.12) \quad &= \left\{ 2 \frac{|\nabla A|^2}{|A|^2} - \frac{|\nabla |A|^2|^2}{|A|^4} + 2 \frac{|\nabla H|^2}{H^2} \right\} \\
 &= \frac{2}{|A|^2} \{ |\nabla A|^2 - |\nabla |A||^2 \}
 \end{aligned}$$

Hence $|\nabla A|^2 = |\nabla |A||^2$ and so $h_{12,k} = 0$, $k = 1, 2$. By the Codazzi equations, $h_{11,2} = h_{22,1} = 0$. Since $h_{22} = r_0 h_{11}$, we deduce $\nabla A = 0$. Thus, M is a complete surface with constant mean curvature. Since M satisfies $H = \langle \nu, e_3 \rangle$, we conclude that M is a plane, a contradiction. Therefore in this case we must have $\frac{G}{F} \equiv 0$, and thus $\kappa_2 \geq 0$.

If G/F achieves its maximum at infinity, by Theorem 2.8 we may apply the Omori-Yau maximum principle and conclude that there exists a sequence P_n tending to infinity with $r_n := \kappa_2/\kappa_1(P_n) \rightarrow r_0$, where $-1 \leq r_0 < 0$. Moreover we have at P_n :

$$\begin{aligned}
 (3.13) \quad \frac{1}{n} &\geq \ddot{\phi}(r_n) \left[\frac{r_n h_{11k}}{\kappa_1} - \frac{h_{22k}}{\kappa_1} \right]^2 + \frac{2}{\kappa_1^2} \dot{\phi}(r_n) \frac{1+r_n}{r_n-1} h_{12k}^2 - 2 \left\langle \frac{\nabla F}{F}, \nabla \left(\frac{G}{F} \right) \right\rangle \\
 &= \ddot{\phi}(r_n) \left[\frac{r_n h_{11k}}{\kappa_1} - \frac{h_{22k}}{\kappa_1} \right]^2 + \frac{2}{\kappa_1^2} \dot{\phi}(r_n) \frac{1+r_n}{r_n-1} h_{12k}^2 - 2 \dot{\phi}(r_n) \left\langle \frac{h_{11k}}{h_{11}}, \frac{h_{22k}}{h_{11}} - r_n \frac{h_{11k}}{h_{11}} \right\rangle
 \end{aligned}$$

and

$$(3.14) \quad C_{n,k} := \frac{h_{22k}}{h_{11}} - r_n \frac{h_{11k}}{h_{11}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Note that

$$\begin{aligned}
 (3.15) \quad \tilde{C}_{n,k} &:= \frac{\nabla_k H}{\kappa_1} = \frac{-\kappa_k \langle \tau_k, e_3 \rangle}{\kappa_1} \\
 &= \frac{h_{22k}}{h_{11}} - r_n \frac{h_{11k}}{h_{11}} + (1+r_n) \frac{h_{11k}}{h_{11}} = C_{n,k} + (1+r_n) \frac{h_{11k}}{h_{11}}
 \end{aligned}$$

From (3.14) we have,

$$(3.16) \quad \frac{h_{22k}}{\kappa_1} = -|r_n| \frac{h_{11k}}{\kappa_1} + C_{n,k}, \quad k = 1, 2$$

and

$$(3.17) \quad \frac{h_{11k} h_{22k}}{\kappa_1^2} = -|r_n| \left(\frac{h_{11k}}{\kappa_1} \right)^2 + C_{n,k} \frac{h_{11k}}{\kappa_1}.$$

Claim: $(1 + r_n) \left| \frac{h_{11k}}{h_{11}} \right| \rightarrow 0 \quad k = 1, 2.$

If not, we can choose a subsequence, still denoted by $\{P_n\}$, so that for $n \geq N$

$$(3.18) \quad (1 + r_n) \left| \frac{h_{112}}{\kappa_1} \right| (P_n) \geq \varepsilon_0, \quad (1 + r_n) \left| \frac{h_{221}}{\kappa_1} \right| (P_n) \geq \varepsilon_0.$$

Then from (3.13) we have

$$(3.19) \quad \begin{aligned} \frac{1}{n} &\geq \ddot{\phi} \sum C_{n,k}^2 - 2\dot{\phi} \sum C_{n,k} \frac{h_{11k}}{\kappa_1} - \frac{2\dot{\phi}}{1 + |r_n|} (1 + r_n) \left(\frac{h_{112}^2}{\kappa_1^2} + \frac{h_{221}^2}{\kappa_1^2} \right) \\ &= \ddot{\phi} \sum C_{n,k}^2 - 2\dot{\phi} \left(C_{n,1} \left(-\frac{1}{|r_n|} \frac{h_{221}}{\kappa_1} + \frac{C_{n,1}}{|r_n|} \right) \right. \\ &\quad \left. + C_{n,2} \frac{h_{112}}{\kappa_1} + \frac{1 + r_n}{1 + |r_n|} \left(\frac{h_{112}^2}{\kappa_1^2} + \frac{h_{221}^2}{\kappa_1^2} \right) \right) \\ &= \ddot{\phi} \sum C_{n,k}^2 - 2\dot{\phi} \frac{C_{n,1}^2}{|r_n|} \\ &\quad - 2\dot{\phi} \left\{ \left(\frac{1 + r_n}{1 + |r_n|} \frac{h_{112}^2}{\kappa_1^2} + C_{n,2} \frac{h_{112}}{\kappa_1} \right) + \left(\frac{1 + r_n}{1 + |r_n|} \frac{h_{221}^2}{\kappa_1^2} - \frac{C_{n,1}}{|r_n|} \frac{h_{221}}{\kappa_1} \right) \right\} \\ &\geq -\frac{2\dot{\phi}}{1 + |r_n|} \left(\left| \frac{h_{112}}{\kappa_1} \right| \left((1 + r_n) \left| \frac{h_{112}}{\kappa_1} \right| - C_{n,2}(1 + |r_n|) \right) \right. \\ &\quad \left. + \left| \frac{h_{221}}{\kappa_1} \right| \left((1 + r_n) \left| \frac{h_{221}}{\kappa_1} \right| - \frac{C_{n,1}}{|r_n|} (1 + |r_n|) \right) \right) \\ &\geq -2 \frac{\dot{\phi}}{1 + |r_n|} \frac{\varepsilon_0}{2} \left(\left| \frac{h_{112}}{\kappa_1} \right| + \left| \frac{h_{221}}{\kappa_1} \right| \right), \end{aligned}$$

which leads to a contradiction for $n \geq N$ by (3.18). Thus the claim is proven and so $\tilde{C}_{n,k} \rightarrow 0$. Therefore $\nu(P_n)$ converges to e_3 and so $H(P_n) \rightarrow 1$.

Now let $\Sigma_n = \Sigma - P_n$ be the surface obtained from Σ by translation of P_n to the origin. Since Σ has bounded principle curvatures, so do the Σ_n . Choosing a subsequence which we still denote by Σ_n , the Σ_n converge smoothly to Σ_∞ . Thus Σ_∞ is a translating soliton which satisfies $H = \langle \nu, e_3 \rangle$, and $H(0) = 1$. Moreover, we have

$$\inf_{x \in \Sigma_\infty} \frac{\kappa_2}{\kappa_1} = \frac{\kappa_2}{\kappa_1}(0).$$

As before we conclude that $G/F = \text{constant}$, and Σ_∞ has constant mean curvature one, which is impossible. Therefore Σ is convex. \square

4. PROOF OF THEOREM 1.4.

In this section we give the proof of

Theorem 4.1. *Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton for the mean curvature flow with positive mean curvature and suppose that $H(P) \rightarrow 0$ as $P \in \Sigma$ tends to infinity. Then Σ is after translation the axisymmetric bowl soliton.*

Proof. Suppose for contradiction that $\Sigma = \text{graph}(u)$ projects onto the strip $|x_1| < R$. Consider for any A , the convex curve $\gamma^A(x_1) = (x_1, A, u(x_1, A)) : -R \leq x_1 \leq R$. Since $u(x_1, A) \rightarrow +\infty$ as $x_1 \rightarrow \pm R$, there is a smallest value $x_1 = x_1(A)$ so that $u_{x_1}(x_1(A), A) = 0$. Since Σ is not a grim cylinder we may assume by a suitable change of coordinates that $u_{x_2}(x_1(0), 0) \geq \delta > 0$. We normalize $u(x_1(0), 0) = 0$; then by convexity of u ,

$$(4.1) \quad \begin{aligned} u(x_1(A), A) &\geq u(x_1(0), 0) + Au_{x_2}(x_1(0), 0) \geq A\delta, \\ 0 = u(x_1(0), 0) &\geq u(x_1(A), A) - Au_{x_2}(x_1(A), A) \geq A(\delta - u_{x_2}(x_1(A), A)), \end{aligned}$$

which implies $u_{x_2}(x_1(A), A) \geq \delta$. By the assumption that $H(P) \rightarrow 0$ as $P \in \Sigma$ tends to infinity, we have that $|\nabla u(x_1(A), A)| = |u_{x_2}(x_1(A), A)| = u_{x_2}(x_1(A), A) \rightarrow \infty$ as $A \rightarrow \infty$. Therefore

$$(4.2) \quad u(x_1, x_2) \geq u(x_1(A), A) + (x_2 - A)u_{x_2}(x_1(A), A) \geq A\delta + (x_2 - A)u_{x_2}(x_1(A), A),$$

and so choosing $x_2 = B$, $A = \frac{B}{2}$,

$$(4.3) \quad \lim_{B \rightarrow \infty} \frac{u(x_1, B)}{B} \geq \frac{\delta}{2} + \frac{1}{2} \lim_{B \rightarrow \infty} u_{x_2}(x_1(\frac{B}{2}), \frac{B}{2}) = +\infty.$$

Hence u grows superlinearly as $x_2 \rightarrow \infty$.

We now compare Σ with a tilted cylinder of radius R . Consider the parametrized family of graphs

$$x_3 = v^t(x_1, x_2) := -\sqrt{1+t^2}\sqrt{R^2 - x_1^2} + t(x_2 - A), \quad |x_1| \leq R, \quad x_2 \geq A$$

of constant mean curvature $H = \frac{1}{R}$ with respect to upward normal direction. Since $v^t(x_1, x_2) \leq t(x_2 - A)$, for any choice of $t \geq 0$, $u > v^t$ for x_2 sufficiently large by (4.3). Also, $u > v^t$ for $|x_1| = R$ and for $x_2 = A$. For $t \leq \delta$, $u > v^t$ in $x_2 \geq A$, $|x_1| \leq R$ by (4.2). Note that $\lim_{t \rightarrow \infty} v^t(x_1, 3A) \geq \lim_{t \rightarrow \infty} (2tA - \sqrt{1+t^2}R) = +\infty$ for $A > R$. We can therefore increase t until there is a first contact of $\Sigma = \text{graph}(u)$ and $\text{graph}(v^t)$, which must occur over an interior point of the half-strip $\{(x_1, x_2), |x_1| < R, x_2 > A\}$.

This gives a $P := (x_1, x_2, u(x_1, x_2)) \in \Sigma$, $x_2 > A$ with $H(P) \geq \frac{1}{R}$. Since A is arbitrary we have a contradiction. \square

5. ASYMPTOTIC BEHAVIOR OF COMPLETE LOCALLY STRICTLY CONVEX TRANSLATING SOLITONS

In this section we study the asymptotic behavior of complete locally strictly convex translating solitons.

Lemma 5.1. *Let $\Sigma = \text{graph}(u)$ be a complete mean convex translating soliton in \mathbb{R}^3 and suppose that the smallest principle curvature κ_2 vanishes at a point of Σ . Then $\kappa_2 \equiv 0$ everywhere and after translation, Σ is grim cylinder of the form $\Sigma = \text{graph}(u^\lambda)$ defined in a strip $\{(x_1, x_2) : |x_1| < R\}$ where*

$$u^\lambda(x_1, x_2) := \lambda^2 \log \sec\left(\frac{x_1}{\lambda}\right) \pm \sqrt{\lambda^2 - 1} x_2, \quad R = \frac{\pi}{2}\lambda, \quad \lambda \geq 1.$$

In particular if Σ contains a line, then Σ is a grim cylinder of the above form.

Proof. If we choose an orthonormal frame τ_1, τ_2 so that $\kappa_2(P) = h_{22}(P)$, then $\kappa_2 \equiv 0 < \kappa_1$ on Σ by Lemma 2.1 part ii. and the maximum principle. Thus the Gauss curvature $K^\Sigma \equiv 0$ and so Σ has a representation (see for example [11]) $\Sigma = \text{graph}(z)$ where $z(x_1, x_2) = \eta(x_1) + \alpha x_2$ for a constant α and a scalar function η defined in a simply connected region containing the projection of P on the x_1, x_2 plane. Therefore

$$(5.1) \quad (1 + \alpha^2)\eta'' = 1 + \alpha^2 + \eta'^2$$

Set $\tilde{\eta}(x_1) = \lambda^{-2}\eta(\lambda x_1)$, where $\lambda^2 = 1 + \alpha^2$. Then

$$(5.2) \quad \tilde{\eta}'' = 1 + \tilde{\eta}'^2$$

and so (up to translation of coordinates and an additive constant)

$$(5.3) \quad \tilde{\eta}(x_1) = \log \sec x_1,$$

which proves the lemma. \square

We have the following Harnack inequality for the mean curvature H on Σ (compare Hamilton [10], Corollary 1.2).

Lemma 5.2. *For any two points $P_1, P_2 \in \Sigma$,*

$$(5.4) \quad H(P_2) \geq e^{-d^\Sigma(P_1, P_2)} H(P_1).$$

Proof. Since $H = \langle N, e_3 \rangle$, $\nabla_k H = -\kappa_k < \tau_k, e_3 \rangle$ so that

$$|\nabla H|^2 \leq |A|^2 = H^2 - 2K \leq H^2.$$

Therefore, $|\nabla \log H| \leq 1$ and (5.4) follows. \square

Lemma 5.3. *Let $\Sigma = \text{graph}(u)$ be a complete mean convex translating soliton in \mathbb{R}^3 defined in the strip $\{(x_1, x_2) : |x_1| < R\}$. Then*

$$H(x_1, x_2) := H((x_1, x_2, u(x_1, x_2))) \leq R - |x_1|.$$

Proof. We use that $\frac{\sum u_{ij}^2}{W^6} \leq |A|^2 \leq H^2 = \frac{1}{W^2}$ where $W^2 = 1 + |\nabla u|^2$. Then $|DW| \leq W^2$ or $|DH| = |D(\frac{1}{W})| \leq 1$. By the convexity of u and the fact that $u(x_1, x_2) \rightarrow \infty$ as $R - |x_1| \rightarrow 0$, x_2 fixed, we see that $H(x_1, x_2) \rightarrow 0$ as $R - |x_1| \rightarrow 0$, x_2 fixed. Hence $H(x_1, x_2) \leq \min(R - x_1, x_1 + R) = R - |x_1|$. \square

Lemma 5.4. *Let $\Sigma = \text{graph}(u)$ be a complete mean convex translating soliton in \mathbb{R}^3 defined over \mathbb{R}^2 . Then $H(P) \rightarrow 0$ for $P \in \Sigma$ tending to infinity.*

Proof. We slightly modify the argument of Haslhofer [12]. Fix $P_0 \in \Sigma$ and suppose there is a sequence $P_n \in \Sigma$ tending to infinity with $\liminf_{n \rightarrow \infty} H(P_n) > 0$. Passing to a subsequence, we may assume $\frac{P_n - P_0}{|P_n - P_0|}$ converges to a unit direction ω . Let $\Sigma_n = \Sigma - P_n$ be the surface obtained from Σ by translation of P_n to the origin. Since Σ has bounded principle curvatures, so do the Σ_n . Choosing a subsequence which we still denote by Σ_n , the Σ_n converge smoothly to Σ_∞ , a convex translating soliton. Moreover, Σ_∞ is an entire graph since the Σ_n are all entire graphs. Since the region K above Σ is convex and $\frac{P_n - P_0}{|P_n - P_0|} \rightarrow \omega$, the limit Σ_∞ contains a line. Therefore by Lemma 5.1, Σ_∞ is a grim cylinder and thus is a graph over a strip, a contradiction. \square

Lemma 5.5. *Let $\Sigma = \text{graph}(u)$ be a complete locally strictly convex translating soliton defined over a strip region $\mathcal{S}^\lambda := \{(x_1, x_2) : |x_1| < R := \lambda \frac{\pi}{2}\}$. Then there exist sequences $P_n = (x_1^n, x_2^n, u(x_1^n, x_2^n))$, $\bar{P}_n = (\bar{x}_1^n, \bar{x}_2^n, u(\bar{x}_1^n, \bar{x}_2^n)) \in \Sigma$ with $x_2^n \rightarrow \infty$, $\bar{x}_2^n \rightarrow -\infty$ and $H(P_n), H(\bar{P}_n) \geq \theta > 0$.*

Proof. By Theorem 1.4 there is a sequence of points $P_n = (x_1^n, x_2^n, u(x_1^n, x_2^n)) \in \Sigma$ where (after relabeling axes if necessary) $\liminf_{n \rightarrow \infty} H(P_n) \geq \theta > 0$ and $x_2^n \rightarrow +\infty$. Note that by Lemma 5.3, $|x_1^n| \leq R - \theta$. We claim there is also a sequence $\bar{P}_n = (\bar{x}_1^n, \bar{x}_2^n, u(\bar{x}_1^n, \bar{x}_2^n)) \in \Sigma$ with $\bar{x}_2^n \rightarrow -\infty$ and $H(\bar{P}_n) \geq \theta > 0$. If not, then for $x_2 \rightarrow -\infty$, $H(x_1, x_2) := H(x_1, x_2, u(x_1, x_2)) \rightarrow 0$. For any $B \leq 0$ let $x_1(B)$ be the smallest

value so that $u_{x_1}(x_1(B), B) = 0$. Since Σ is not a grim cylinder, we may assume by a translation of coordinates that $u_{x_2}(x_1(0), 0) = \delta > 0$. We normalize $u(x_1(0), 0) = 0$; then

$$u(x_1(B), B) \geq u(x_1(0), 0) + Bu_{x_2}(x_1(0), 0) = B\delta,$$

and

$$0 = u(x_1(0), 0) \geq u(x_1(B), B) - Bu_{x_2}(x_1(B), B) \geq B(\delta - u_{x_2}(x_1(B), B)).$$

Hence $u_{x_2}(x_1(B), B) \leq \delta$. Since $H(x_1, x_2) \rightarrow 0$ as $x_2 \rightarrow -\infty$, we conclude

$$(5.5) \quad u_{x_2}(x_1(B), B) \rightarrow -\infty \text{ as } B \rightarrow -\infty.$$

Therefore

$$(5.6) \quad u(x_1, x_2) \geq u(x_1(B), B) + (x_2 - B)u_{x_2}(x_1(B), B) \geq B\delta + (x_2 - B)u_{x_2}(x_1(B), B)$$

Now choose $x_2 = \Lambda < 0$ and $B = \frac{\Lambda}{2}$. Then

$$u(x_1, \Lambda) \geq \frac{\delta\Lambda}{2} + \frac{\Lambda}{2}u_{x_2}(x_1(\frac{\Lambda}{2}), \frac{\Lambda}{2}),$$

hence

$$(5.7) \quad \lim_{\Lambda \rightarrow -\infty} \frac{u(x_1, \Lambda)}{\Lambda} \leq \frac{\delta}{2} + \frac{1}{2}u_{x_2}(x_1(\frac{\Lambda}{2}), \frac{\Lambda}{2}) \rightarrow -\infty$$

by (5.5). Thus $u(x_1, \Lambda) \rightarrow \infty$ superlinearly as $\Lambda \rightarrow -\infty$. We now compare Σ with a tilted cylinder of radius R . Consider $x_3 = v^t(x_1, x_2) := -\sqrt{1+t^2}\sqrt{R^2-x_1^2} + t(x_2-B)$ in the half-strip $\mathcal{S}^B := \{(x_1, x_2) : |x_1| \leq R, x_2 \leq B < 0\}$. Since $v^t \leq t(x_2-B)$, for any choice of $t \leq 0$, $u > v^t$ for x_2 sufficiently small (i.e. x_2 large negative) by (5.7). Also $u > v^t$ for $|x_1| = R$ and $x_2 = B < 0$, as soon as $u(x_1, B) > 0$. For $t \geq -\delta$, $u > v^t$ in \mathcal{S}^B and since

$$\lim_{t \rightarrow -\infty} v^t(x_1, 3B) \geq \lim_{t \rightarrow -\infty} (2Bt - \sqrt{1+t^2}R) \rightarrow \infty,$$

we can decrease t until there is a first contact point $P \in \mathcal{S}^B$ of $\text{graph}(v^t)$ and Σ . At $P \in \Sigma$, $H(P) \geq \frac{1}{R}$, a contradiction. □

We now analyze the asymptotic behavior of complete locally strictly convex translating solitons $\Sigma = \text{graph}(u)$ in \mathbb{R}^3 that are defined over the strip region \mathcal{S}^λ .

Theorem 5.6. *Let $\Sigma = \text{graph}(u)$ be a complete locally strictly convex translating soliton defined over a strip region $\mathcal{S}^\lambda := \{(x_1, x_2) : |x_1| < R := \lambda\frac{\pi}{2}, \lambda \geq 1\}$. Then (after possibly relabeling the e_2 direction)*

i. For $\lambda \leq 1$ there is no locally strictly convex solution in \mathcal{S}^λ .

$$ii. \lim_{x_2 \rightarrow +\infty} u_{x_2}(x_1, x_2) = L := \sqrt{\lambda^2 - 1}, \lambda > 1.$$

$$iii. \lim_{x_2 \rightarrow -\infty} u_{x_2}(x_1, x_2) = -L.$$

iv.

$$(5.8) \quad \begin{aligned} \lim_{A \rightarrow \pm\infty} (u(x_1, x_2 + A) - u(0, A)) &= u^\lambda(x_1, x_2) \\ \lim_{A \rightarrow \pm\infty} u_{x_1}(x_1, A) &= \lambda \tan \frac{x_1}{\lambda}. \end{aligned}$$

The above limits hold uniformly on $\mathcal{S}^\varepsilon = \{(x_1, x_2) \in \mathcal{S}^\lambda : |x_1| \leq R - \varepsilon\}$.

v. For $P = (x_1, x_2, u(x_1, x_2)) \in \Sigma$, $(x_1, x_2) \in \mathcal{S}^\varepsilon$,

$$(5.9) \quad H(P) \geq \theta(\varepsilon).$$

Proof. We will prove parts i,ii; the argument for iii. is the same by Lemma 5.5. By Lemma 5.5 there is a sequence of points $P_n \in \Sigma$ where (after relabeling axes if necessary) $\liminf_{n \rightarrow \infty} H(P_n) \geq \theta > 0$ and $x_2^n \rightarrow +\infty$. Note that by Lemma 5.3, $|x_1^n| \leq R - \theta$. Arguing as in lemma 5.4, we find as above Σ_n converge smoothly to Σ_∞ , a convex translating soliton defined over the strip $\mathcal{S}^\lambda - (x_1^\infty, 0)$ passing through the origin. By Lemma 5.1, $\lambda > 1$ and $u(y_1 + x_1^n, y_2 + x_2^n) - u(x_1^n, x_2^n)$ converges smoothly to

$$\lambda^2 \log \sec \left(\frac{y_1 + x_1^\infty}{\lambda} \right) + \sqrt{\lambda^2 - 1} y_2 - \lambda^2 \log \sec \frac{x_1^\infty}{\lambda}$$

as $n \rightarrow \infty$. In particular $u_{x_2}(y_1 + x_1^n, y_2 + x_2^n)$ converges smoothly to $\sqrt{\lambda^2 - 1}$ as $n \rightarrow \infty$. Replacing y_1 by $x_1 - x_1^\infty$ we find $\limsup_{x_2 \rightarrow \infty} u_{x_2}(x_1, x_2) \geq \sqrt{\lambda^2 - 1}$. On the other hand, $u_{x_2}(x_1, x_2) \leq u_{x_2}(x_1, x_2 + x_2^n)$ for n large so parts i,ii iii. are proven.

We will prove iv. for $x_2 \rightarrow +\infty$ as the argument in the other case is identical. Note that by part ii, for $x_2 = A$, $x_1 = x_1(A)$ (recall $u_{x_1}(x_1(A), A) = 0$), $H((x_1(A), A, u(x_1(A), A))) \geq \frac{1}{\sqrt{1+L^2}}$. Hence we may choose $x_2^n \rightarrow \infty$ arbitrary and $x_1^n = x_1(x_2^n)$. Arguing as in part ii., we have

$$(5.10) \quad \begin{aligned} i. \quad \lim_{n \rightarrow \infty} (u(y_1 + x_1^n, y_2 + x_2^n) - u(x_1^n, x_2^n)) &= u^\lambda(y_1 + x_1^\infty, y_2) - \lambda^2 \log \sec \frac{x_1^\infty}{\lambda}, \\ ii. \quad \lim_{n \rightarrow \infty} u_{x_1}(y_1 + x_1^n, y_2 + x_2^n) &= \lambda \tan \frac{(y_1 + x_1^\infty)}{\lambda}. \end{aligned}$$

in $|y_1 + x_1^\infty| < R$. Setting $y_1 = y_2 = 0$ in (5.10) ii., we conclude $x_1^\infty = 0$.

We claim that

$$(5.11) \quad \begin{aligned} i. \quad \lim_{n \rightarrow \infty} (u(x_1, x_2 + x_2^n) - u(0, x_2^n)) &= u^\lambda(x_1, x_2), \\ ii. \quad \lim_{n \rightarrow \infty} u_{x_1}(x_1, x_2 + x_2^n) &= \lambda \tan \frac{x_1}{\lambda} \end{aligned}$$

in \mathcal{S} . Since $|D(1/W)| \leq 1$,

$$|1/W(x_1 + x_1^n, x_2 + x_2^n) - 1/W(x_1, x_2 + x_2^n)| \leq |x_1^n|,$$

and so by (5.10) ii. (since $\lim_{x_2 \rightarrow \infty} u_{x_2}(x_1, x_2) = L$),

$$(5.12) \quad \lim_{n \rightarrow \infty} W(x_1, x_2 + x_2^n) = \lambda \sec \frac{x_1}{\lambda}.$$

Using $\sum |u_{ij}(x_1, x_2^n)| \leq W^2(x_1, x_2^n)$ (as in Lemma 5.3), we have by (5.12)

$$(5.13) \quad \begin{aligned} & |u_1(x_1 + x_1^n, x_2 + x_2^n) - u_1(x_1, x_2 + x_2^n)| \leq |x_1^n| |u_{11}(x_1 + O(|x_1^n|), x_2 + x_2^n)| \\ & \leq |x_1^n| W^2(x_1 + O(x_1^n), x_2^n) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Therefore we obtain

$$\lim_{n \rightarrow \infty} u_{x_1}(x_1, x_2 + x_2^n) \rightarrow \lambda \tan \frac{x_1}{\lambda},$$

proving (5.11) ii. Now since

$$|u(x_1 + x_1^n, x_2 + x_2^n) - u(x_1, x_2 + x_2^n)| \leq |x_1^n| |u_{x_1}(x_1 + O(|x_1^n|), x_2 + x_2^n)| \rightarrow 0,$$

and

$$|u(x_1^n, x_2^n) - u(0, x_2^n)| \leq |x_1^n| |u_{x_1}(O(|x_1^n|), x_2^n)| \rightarrow 0 \text{ by (5.11) ii},$$

(5.11) i. follows from (5.10) i. Since the choice of x_2^n is arbitrary, part iv. follows.

To prove part v. we will use Lemma 5.2 with

$$P_2 = (x_1, x_2, u(x_1, x_2)), P_1 = (x_1(x_2), x_2, u(x_1(x_2), x_2)), H(P_1) \geq \frac{1}{\sqrt{1+L^2}}.$$

Normalize u by $|\nabla u(x_1(0), 0)| = 0$. We observe that

$$d^\Sigma(P_2, P_1) \leq L(x_2) := \int_{-R+\varepsilon}^{R-\varepsilon} \sqrt{1 + u_{x_1}^2(t, x_2)} dt$$

and by (5.8), $L(x_2) \leq M(\varepsilon)$ for $|x_2| \geq A(\varepsilon)$ sufficiently large. Since u is a convex solution to the translating soliton equation,

$$(5.14) \quad \begin{aligned} & (1 + u_{x_2}^2)u_{x_1x_1} + (1 + u_{x_1}^2)u_{x_2x_2} = 2u_{x_1}u_{x_2}u_{x_1x_2} + (1 + u_{x_1}^2 + u_{x_2}^2) \\ & \leq 2|u_{x_1}||u_{x_2}|\sqrt{u_{x_1x_1}u_{x_2x_2}} + (1 + u_{x_1}^2 + u_{x_2}^2) \\ & \leq (1 + u_{x_1}^2)u_{x_2x_2} + \frac{u_{x_1}^2 u_{x_2}^2}{(1 + u_{x_1}^2)}u_{x_1x_1} + (1 + u_{x_1}^2 + u_{x_2}^2). \end{aligned}$$

This implies

$$(5.15) \quad \begin{aligned} & u_{x_1x_1} \leq 1 + u_{x_1}^2 \text{ and by symmetry } u_{x_2x_2} \leq 1 + u_{x_2}^2 \leq 1 + L^2 \\ & |u_{x_1x_2}| \leq \sqrt{u_{x_1x_1}u_{x_2x_2}} \leq \sqrt{1 + L^2} \sqrt{1 + u_{x_1}^2}. \end{aligned}$$

Therefore using (5.15), $|L'(x_2)| \leq \sqrt{1+L^2} L(x_2)$ and so

$$(5.16) \quad d^\Sigma(P_2, P_1) \leq L(x_2) \leq e^{\sqrt{1+L^2}(A(\varepsilon)-|x_2|)} M(\varepsilon) \leq e^{\sqrt{1+L^2}A(\varepsilon)} M(\varepsilon) =: \overline{M}(\varepsilon)$$

for $|x_2| \leq A(\varepsilon)$. Therefore $H(P_1) \geq \theta(\varepsilon) := \frac{e^{-\overline{M}(\varepsilon)}}{\sqrt{1+L^2}}$, completing the proof. \square

An immediate application of Theorem 5.6 is the following symmetry result.

Theorem 5.7. *Let $\Sigma = \text{graph}(u)$ be a complete locally strictly convex translating soliton defined over a strip region \mathcal{S}^λ . Then $u(x_1, x_2) = u(-x_1, x_2)$ and $u_{x_1}(x_1, x_2) > 0$ for $x_1 > 0$.*

Proof. We employ the method of moving planes; see for example [2]. Set $\mathcal{S}_t = \{(x_1, x_2) \in \mathcal{S}^\lambda : t < x_1 < R\}$ for $t \in (0, R)$. We want to show that the function

$$(5.17) \quad v^t(x_1, x_2) := u(x_1, x_2) - u(2t - x_1, x_2) > 0 \text{ in } \mathcal{S}_t \text{ for all } t \in (0, R).$$

Once (5.17) is proven, the conclusion of Theorem 5.7 follows easily. Indeed letting $t \rightarrow 0$ in (5.17) implies by continuity

$$(5.18) \quad u(x_1, x_2) \geq u(-x_1, x_2) \text{ for } 0 \leq x_1 \leq R.$$

Since we may replace x_1 by $-x_1$ in (5.18), we have equality and thus we have symmetry in x_1 . From (5.17) we may also conclude that $u_{x_1} \geq 0$ for $0 < x_1 < R$. Since u_{x_1} satisfies an elliptic equation without zeroth order term, we have from the maximum principle either $u_{x_1} > 0$ or $u_{x_1} \equiv 0$. Since the latter possibility is impossible, Theorem 5.7 follows.

Set $u^t = u(2t - x_1, x_2)$. Since u and u^t both satisfy the same elliptic equation (1.4) in \mathcal{S}_t , the difference $v^t = u - u^t$ satisfies a linear elliptic equation

$$(5.19) \quad \sum_{i,j=1}^2 A^{ij} v_{ij}^t + \sum_{i=1}^2 B^i v_i^t = 0 \text{ in } \mathcal{S}_t,$$

and so v^t cannot have an interior minimum in \mathcal{S}_t . Also $v^t = 0$ for $x_1 = t$, $\lim_{x_1 \rightarrow R} v^t = +\infty$, and by Theorem 5.6 part iv.,

$$(5.20) \quad \lim_{x_2 \rightarrow \pm\infty} v^t = \lambda^2 \log \left\{ \frac{\sec \frac{x_1}{\lambda}}{\sec \frac{(2t-x_1)}{\lambda}} \right\} \geq 0.$$

Therefore $v_t \geq 0$ and so by the maximum principle (5.17) holds completing the proof. \square

6. EXISTENCE OF A COMPLETE LOCALLY STRICTLY CONVEX TRANSLATING SOLITON
 $\Sigma = \text{GRAPH}(w)$ IN THE STRIP \mathcal{S}^λ

In this section we prove existence of a complete locally strictly convex translating solitons in a strip and show there are no other locally strictly convex solutions.

Theorem 6.1. *There exists a doubly symmetric complete translating soliton $\Sigma = \text{graph}(w)$ in \mathcal{S}^λ , $\lambda > 1$.*

Before we start our construction, we recall a well known compactness theorem (see for example [17]) for the mean curvature flow.

Proposition 6.2. *(compactness of mean curvature flow) Let $\{(\Sigma_i^n, X_i(t)), -1 < t < 1\}$ be a sequence of mean curvature flow properly immersed in $B_\rho(0) \subset \mathbb{R}^{n+1}$. Suppose that*

$$\sup_{\Sigma_{i,t} \cap B_\rho(0)} |A|(x, t) \leq \Lambda, \quad \forall t \in (-1, 1)$$

for some $\Lambda > 0$. Then a subsequence of $\{\Sigma_{i,t} \cap B_\rho(0), -1 < t < 1\}$ converges in the C^∞ topology to a smooth mean curvature flow $\{\Sigma_{\infty,t}, -1 < t < 1\}$ in $B_\rho(0)$.

Consider the family of rectangles

$$R_A = \{x \in \mathbb{R}^2 : |x_1| < R, |x_2| < A, R = \lambda \frac{\pi}{2}, \lambda > 1, A \geq 1\}.$$

and let u_A be the solution of the Dirichlet problem

$$(6.1) \quad \begin{cases} \sum_{i,j} \left(\delta_{ij} - \frac{u_i u_j}{1 + |Du|^2} \right) u_{ij} = 1 & \text{in } R_A, \\ u = 0 & \text{on } \partial R_A. \end{cases}$$

The existence of a unique smooth solution in R_A is standard (we can round the corners to form a smooth convex doubly symmetric domain but this is not essential). Moreover by the method of moving planes, u_A is symmetric in both x_1, x_2 with its negative minimum at $(0,0)$. Let $M_A = -\inf u_A$; then by the maximum principle M_A is positive and strictly increasing.

Lemma 6.3. $M_A \rightarrow \infty$ as $A \rightarrow \infty$.

Proof. The proof is by contradiction. Assume $M_A \rightarrow C$ as $A \rightarrow \infty$. Then we have $u_A \rightarrow u$ (since the family $u_A + M_A$ is monotone increasing with all derivatives uniformly bounded) and u is a solution of (6.1) in \mathcal{S}^λ . Moreover, we have $\inf u = -C$ and $u = 0$ on $\partial \mathcal{S}$. Let $v(x) = \log \sec x_1$ be the graph of the (standard) grim cylinder in $|x_1| < \frac{\pi}{2}$.

Then $v(x) > u(x, y)$ and $C_1 = \inf(v(x) - u(x, y))$ is bounded. Now let $\tilde{u}^n(x, y) = u(x, y + y_n) + C_1$ where $(v(x_n) - u(x_n, y_n)) \rightarrow C_1$. By the maximum principle we must have $|(x_n, y_n)| \rightarrow \infty$. Assume $x_n \rightarrow x_0$ and note $|x_0| < \frac{\pi}{2}$. Since u is bounded and $u = \text{constant}$ on the boundary of the strip, all derivative of \tilde{u} are uniformly bounded. Hence a subsequence of \tilde{u}^n converges smoothly to a solution $u^\infty(x, y) \leq v(x)$ with equality at $(x_0, 0)$. This contradicts the maximum principle. \square

A direct consequence of Lemma 6.3 is the following

Lemma 6.4. *For any $k > 1$, there exist $A = A(k)$ such that $M_{A(k)} = k$.*

Now let $\{u_{A_k}\}$ be a sequence of solution of equation (6.1), such that $\inf u_{A_k} = -k$. Denote $w_k = u_{A_k} + k$; then $w_k \geq w_k(0) = 0$. We claim that a subsequence of the w_k converges to a solution $\Sigma = \text{graph}(w)$ defined on the strip region S^λ . This follows from Propositions 2.4, 2.7, and the Compactness Theorem, Proposition 6.2.

It remains show that $\Sigma = \text{graph}(w)$ is complete.

Lemma 6.5. *For any $x_0 \in \partial S^\lambda$ we have*

$$\lim_{x \rightarrow x_0} w(x) = \infty.$$

Proof. We argue by contradiction. Fix $x_0 \in \partial S^\lambda$ and suppose there is a sequence $x_n \in S^\lambda$ such that $x_n \rightarrow x_0$ and $w(x_n) \leq C$. Then for fixed $k > C$ such that x_0 is in the domain of w_k , $w_k(x_n) \leq w(x_n) \leq C$. But w_k is continuous at x_0 so $k = \lim_n w_k(x_n) \leq C$, a contradiction. \square

By Theorem 1.1, $\Sigma = \text{graph}(w)$ is convex and since $w \geq 0$, $w \neq u^\lambda$ so w is locally strictly convex, completing the proof of Theorem 5 part vii.

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